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Modelling Bond Duration and Convexity Under Interest Rate and Time to Maturity Variations

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Abstract

Any bond, or a portfolio of bonds, can be synthesized with its yield to maturity, duration, and convexity. A single zero-coupon bond can replicate the yield, and the duration of a bond, but not its convexity. Two zero-coupon bonds are required to model the yield, duration, and convexity of a bond - or a portfolio of bonds. First, this paper derives the parameters of the two zero-coupon bonds to model a bond, or bond portfolio. Then a comparison is made of different methods to estimate the variation of a bond value, under interest rate, and time to maturity variations. The conclusion is that a single zero-coupon bond, even though it does not replicate the convexity of a vanilla bond, or a bond portfolio, is a simple and accurate model to estimate bond values under interest rate and time to maturity changes. Other factors that influence the price of the bond such as default risk, call risk, liquidity, etc have not been incorporated in the model.

I Background

The value of an asset is the present value of the cash flows expected from the asset.

A standard bond ("a vanilla bond") pays periodic coupons, normally one per semester, and at the time of maturity it returns the nominal value of the bond. Therefore, the value of a bond would be:

$$V = \left(\sum_{j=1}^{N} \frac{P_j}{(1+i)^j}\right) + \frac{P_N + M}{(1+i)^N} \tag{1}$$

where:

P_j = Payment at time j
i = Discount rate per period
M = Nominal value of the bond
N = Number of periods of the bond
j = period

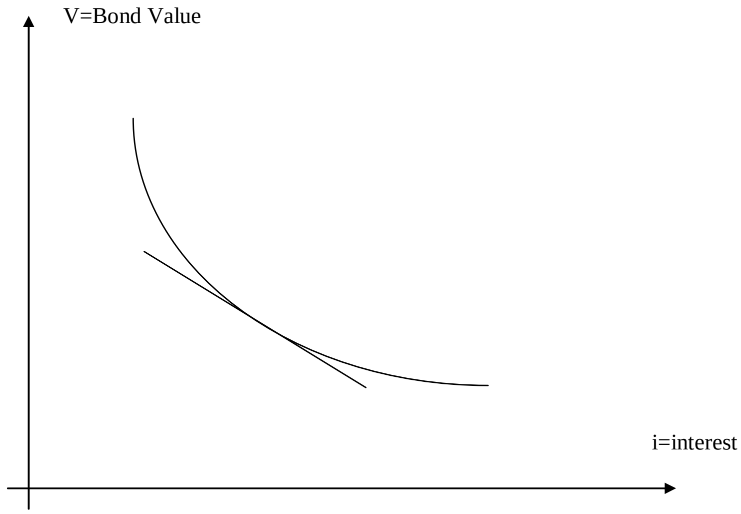
If we define w_j as the per unit weight of the total value, corresponding to the payment in period j, then the "Macauley Duration" D can be obtained through:

$$D = \sum_{j=1}^{N} j w_j \tag{2}$$

Modified Duration is:

$$MD = \frac{-D}{(1+i)} \tag{3}$$

Modified Duration is important because it is the first derivative of the value of the bond V, as a function of the interest rate i.



Therefore, knowing the Modified Duration, the variation of the value of the bond can be estimated for a small variation of the interest rate.

$$\Delta V \cong MD\Delta i \qquad (4)$$

As the graph shows, due to the convexity of the curve, for large variations of the interest rate the Modified Duration is not a good estimate. The convexity is a correction to estimate more accurately the value of the bond when large variations of the interest rate occur.

$$\text{Convexity} = \text{C} = \frac{\sum_{j=1}^{N}(j + j^2)w_j}{(1+i)^2} \qquad (5)$$

Using a Taylor Series expansion, for a variation of the interest rate, the variation in the value of the bond can be estimated from:

$$\Delta V \cong (-MD\Delta i) + \frac{C(\Delta i)^2}{2} \qquad (6)$$

II Zero-Coupon Bond

The value of a zero coupon bond is:

$$V = \frac{P}{(1+i)^N} \qquad (7)$$

The Macaulay Duration of a zero-coupon bond is equal to its time to maturity.

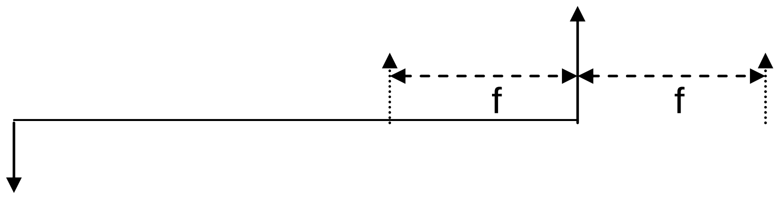
The convexity of a zero-coupon bond is then:

$$C = \frac{D + D^2}{(1+i)^2} \qquad (8)$$

III Modelling the Bond

A single zero-coupon bond can replicate the Value, and the Duration of a vanilla bond.

To replicate the convexity, two zero-coupon bonds are required.



Instead of a single zero-coupon bond, we will have two, of equal present value, and separated a "distance" in time of 2f.

The convexity increases with the separation f.

The process can be visualized as separating the two zero-coupon bonds (increasing f) until the Convexity of the bond to be replicated is reached.

Nonetheless, the distance f can be calculated very simply.

$$C(1+i)^2 = \left[(D+f)(D+f)^2\right]^{-}\left[(D-f)+(D-f)^2\right] \tag{9}$$

From which we obtain:

$$f = \left[(1+i)^2 C - D^2 - D\right]^{1/2} \tag{10}$$

IV. Numerical Example of Estimates of Bond Value under Interest Rate Change

Let us say that we need to replicate a three-year vanilla bond with a yield of 8 %. The bond has a nominal value of $1000 and it pays 6 semi-annual coupons of $40 each.

Period	Payment	Present Worth	Weight	Duration	Convexity
0	-$1000				
1	$40	$38.4615	0.03846	0.03846	0.07692
2	$40	$36.9822	0.03698	0.07396	0.22189
3	$40	$35.5599	0.03556	0.10668	0.42672
4	$40	$34.1922	0.03419	0.13677	0.68384
5	$40	$32.8771	0.03288	0.16439	0.98631
6	$1040	$821.9271	0.82193	4.93156	34.5209
	SUM	$1000.0000		5.45182	36.9166
		Modified Duration=		5.2421	
		Convexity =			34.1315

The period here is one semester.

The annualized Modified Duration would be:

$MD_{Annualized} = 5.2421 / 2 = 2.6211$ years

The annualized convexity would be:

$\text{Convexity}_{Annualized} = 34.1315 / 4 = 8.5329$

Applying formula (10) we obtain:

f = 1.3200

The two zero-coupon bonds that replicate this vanilla bond are:

Bond A: Maturity= Macaulay Duration =5.4518-1.32 =4.1318 periods Value at Maturity = $587.96

Bond B: Maturity= Macaulay Duration =5.4518+1.32 =6.7718 periods Value at Maturity = $652.10

A single zero-coupon bond, let us call it S, would have a time of maturity of 5.4518 periods, and a value at maturity of $1238.41. As has been indicated before, this bond would not replicate the convexity of the vanilla bond.

We now have several alternatives to calculate the vanilla bond value, under interest rate changes.

Let us say that the annual interest rate is now 12 %.

- True value = $901.65
- Two zero-coupon bonds (A and B)= $462.16 + $439.49 = $901.65
- Modified Duration Formula = $1000 x (1 + (-2.6211 x 0.04)) = $895.16
- Modified Duration + Convexity Formula = $895.16 + (1+(0.04 x0.04 x 8.5329 / 2)) = $895.16 + $6.83 = $901.98
- One zero-coupon bond = $901.36

The differences, except for the duration formula, are insignificant.

V. Numerical Example of Estimates of Bond Value under Interest Rate and Time to Maturity Change

If we want to calculate the value of a bond after a period of time has elapsed, the Modified Duration and the Modified Duration + Convexity Formulae do not apply.

Let us say, that we need to estimate the value of the bond of Section IV, three months from now.

We now have:
True value of the Bond= $943.15
Two zero-coupon bonds (A and B)= $943.14
One zero-coupon bond = $942.75

VI. Conclusions

The procedure indicated in this paper has been applied to calculate the value of short and long term bonds, under interest rate and time to maturity variations. The conclusion is that for practical purposes the single zero-coupon bond provides the required accuracy, and simplicity, to model bonds under interest rate and time to maturity variations.

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